Mathematical annex to “A Generalised Variable Elasticity of Substitution Model of New Economic Geography”

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Abstract

This annex contains the detailed derivations of the equations in the manuscript, in order to facilitate the evaluation of its mathematical content. For purpose of clarity and to facilitate comparison with the equations in the manuscript, the general layout of this annex follows the structure of the manuscript.

1 Adapting the flexible variety aggregator to NEG

1.1 Sub-utility

The size of the manufacturing aggregate is (trivially) given by the overall utility maximisation problem.

$$M_i = \frac{w_i L_i}{G_i} \tag{1}$$

The separable sub-utility problem involves choosing the combination of manufacturing varieties that minimises the manufacturing expenditure $M_i G_i$ given in (2):

$$M_i G_i = \int_0^N p_{*,i}(h)\, \tau_{*,i} m_{*,i}(h)\, dh \tag{2}$$

$$\begin{cases} \min & \int_0^N p_{*,i}(h)\,\tau_{*,i} m_{*,i}(h)\,dh \\ \text{s.t.} & \int_0^{\text{N}} \varphi\left(\frac{m_{*,i}(h)}{M_i}\right) dh = 1 \end{cases} \tag{3}$$

Setting up the Lagrangian gives the following first order conditions:

$$\Lambda_i = \int_0^N p_{*,i}(h)\,\tau_{*,i} m_{*,i}(h)\,dh - \lambda\left(\int_0^N \varphi\left(\frac{m_{*,i}(h)}{M_i}\right) dh - 1\right) \tag{4}$$

$$\begin{cases} \frac{\partial \Lambda_i}{\partial m_{*,i}(h)} = 0 & \Rightarrow \quad p_{*,i}(h)\,\tau_{*,i} = \frac{\lambda}{M_i}\varphi'\left(\frac{m_{*,i}(h)}{M_i}\right) \\ \frac{\partial \Lambda_i}{\partial \lambda} = 0 & \Rightarrow \quad \int_o^N \varphi\left(\frac{m_{*,i}(h)}{M_i}\right) dh = 1 \end{cases} \tag{5}$$

Replacing the star notation by the standard i, j notation and inverting the first order condition (5) gives the compensated demand for a variety consumed in region j and produced in i. Note that this compensated demand is also the manufacturing share variable $s_{i,j}(h)$.

$$s_{i,j}(h) = \frac{m_{i,j}(h)}{M_j} = \varphi'^{-1}\left(\frac{p_{i,j}(h)\,\tau_{i,j}}{\tilde{P}_j}\right) \tag{6}$$

In terms of notation, superscript -1 indicates an inverse function, with the compositional price index $\tilde{P}_i$ defined as:

$$\tilde{P}_i = \frac{\lambda}{M_i} \tag{7}$$

This price index $\tilde{P}_i$ can be specified completely once a functional form for $\varphi(x)$ is chosen by replacing the first order conditions into the constraint.

A formal definition of the manufacturing price index G_i can be obtained by replacing (6) in the expenditure equation (2):

$$M_i G_i = \int_0^N p_{*,i}(h)\,\tau_{*,i} m_{*,i}(h)\,dh$$

$$M_i G_i = \int_0^N p_{*,i}(h)\,\tau_{*,i}\varphi'^{-1}\left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\tilde{P}_i}\right) M_i dh$$

$$G_i = \int_0^N p_{*,i}(h)\,\tau_{*,i}\varphi'^{-1}\left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\tilde{P}_i}\right) dh \tag{8}$$

As for the compositional price index $\tilde{P}_i$, this definition is dependant on the choice of specification for the sub-utility function $\varphi(x)$.

1.2 Pricing behaviour of firms

As stated in the manuscript, production of a manufacturing variety is assumed to use only labour, with a fixed cost α and a variable cost β. The total cost of production for all the flows is given by:

$$C_i(h) = w_i\left(\alpha + \beta\sum_j m_{i,j}(h)\right) \tag{9}$$

The profit made by a producer in region i is:

$$\pi_i(h) = \sum_j p_{i,j}(h)\,m_{i,j}(h) - C_i(h) \tag{10}$$

Manufacturing profits must be maximised for all flows, which gives the familiar Lerner index first order conditions for prices.

$$\frac{\partial \pi_i(h)}{\partial m_{i,j}(h)} = \frac{\partial p_{i,j}(h)}{\partial m_{i,j}(h)} m_{i,j}(h) + p_{i,j}(h) - \beta w_i = 0$$

$$p_{i,j}(h)\left(\frac{\partial p_{i,j}(h)}{\partial m_{i,j}(h)}\frac{m_{i,j}(h)}{p_{i,j}(h)} + 1\right) = \beta w_i \tag{11}$$

The key element here is the price elasticity of demand in a given region, which we shall denote $\varepsilon_{i,j}$.

$$\frac{1}{\varepsilon_{i,j}(h)} = \frac{\partial p_{i,j}(h)}{\partial m_{i,j}(h)}\frac{m_{i,j}(h)}{p_{i,j}(h)}$$

Its value can be calculated using the definition for $p_{i,j}(h)$ provided in the first order condition (5).

$$\frac{1}{\varepsilon_{i,j}(h)} = \frac{\tilde{P}_j}{M_j \tau_{i,j}}\varphi''\left(\frac{m_{i,j}(h)}{M_j}\right) \times \frac{m_{i,j}(h)}{\frac{\tilde{P}_j}{\tau_{i,j}}\varphi'\left(\frac{m_{i,j}(h)}{M_j}\right)}$$

$$\frac{1}{\varepsilon_{i,j}(h)} = \frac{m_{i,j}(h)}{M_j} \frac{\varphi''\left(\frac{m_{i,j}(h)}{M_j}\right)}{\varphi'\left(\frac{m_{i,*}(h)}{M_j}\right)} \tag{12}$$

In order to simplify the notation, we use here the share variable $s_{i,j}(h)$ defined in (6):

$$\frac{1}{\varepsilon_{i,j}(h)} = s_{i,j}(h) \frac{\varphi''(s_{i,j}(h))}{\varphi'(s_{i,j}(h))} \tag{13}$$

Which is the Arrow-Pratt measure of relative risk-aversion found in the manuscript.

1.3 The role of the sub-utility function

The sub-utility function $\varphi(x)$ given in the manuscript is:

$$\varphi(x) = \frac{(\eta x - (\eta - 1))^{\rho}}{\eta} - \frac{(1-\eta)^{\rho}}{\eta} \tag{14}$$

with $\rho = (\sigma - 1)/\sigma$ if one wants a specification similar to the standard Dixit-Stiglitz model. Given the chosen specification for the sub-utility function, this implies the following derivatives and inverses:

First derivative:

$$\varphi'(x) = \rho (\eta x - (\eta - 1))^{\rho - 1} \tag{15}$$

One can see that $\varphi'(x) > 0$ for $0 < \rho < 1$ and $0 < \eta \leq 1$

Second derivative:

$$\varphi''(x) = \eta\rho(\rho - 1)(\eta x - (\eta - 1))^{\rho - 2} \tag{16}$$

One can see that $\varphi''(x) < 0$ for $0 < \rho < 1$ and $0 < \eta \leq 1$

Inverse of the first derivative:

$$\varphi'^{-1}(x) = \frac{1}{\eta}\left(\left(\frac{x}{\rho}\right)^{\frac{1}{\rho-1}} + (\eta - 1)\right) \tag{17}$$

1.4 Solving for prices, quantities and mark-ups

Intermediate Price index $\tilde{P}_i$

The first step is the determination of the intermediate price index $\tilde{P}_i$, by replacing the share variable (6) into the implicit definition of the aggregator in (3). This allows a definition of $\tilde{P}_i$ as a price index rather than the initial Lagrange multiplier based definition in equation (7).

$$\int_0^N \varphi\left(\varphi'^{-1}\left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\tilde{P}_i}\right)\right) dh = 1$$

Given the specifications of $\varphi(x)$ given in (14) and (17), one can work out the integral in the previous equation.

$$\int_0^N \varphi\left(\frac{1}{\eta}\left(\left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\rho\tilde{P}_i}\right)^{\frac{1}{\rho-1}} + (\eta-1)\right)\right) dh = 1$$

$$\int_0^N \frac{1}{\eta}\left(\left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\rho\tilde{P}_i}\right)^{\frac{\rho}{\rho-1}} - (1-\eta)^{\rho}\right) dh = 1$$

$$\int_0^N \left(\frac{p_{*,i}(h)\,\tau_{*,i}}{\tilde{P}_i}\right)^{\frac{\rho}{\rho-1}} dh = \rho^{\frac{\rho}{\rho-1}}\left(\eta + N(1-\eta)^{\rho}\right)$$

$$\tilde{P}_i = \frac{\left(\int_0^N \left(p_{*,i}(h)\,\tau_{*,i}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{\rho-1}{\rho}}}{\rho\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{\rho-1}{\rho}}} \tag{18}$$

Share variable $s_{i,j}(h)$

From the specification of $\tilde{P}_i$ in (18), one can obtain for each variety the $(p_{i,j}(h)\tau_{i,j})/\tilde{P}_j$ ratio, which can then be used in equation (6) to determine manufacturing share variable $s_{i,j}(h)$ in each region.

$$\frac{p_{i,j}(h)\,\tau_{i,j}}{\tilde{P}_j} = p_{i,j}(h)\,\tau_{i,j}\frac{\rho\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{\rho-1}{\rho}}}{\left(\int_0^N \left(p_{*,j}(h)\,\tau_{*,j}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{\rho-1}{\rho}}} \tag{19}$$

Replacing in (6):

$$s_{i,j}(h) = \varphi'^{-1}\left(\frac{\rho\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{\rho-1}{\rho}} p_{i,j}(h)\,\tau_{i,j}}{\left(\int\limits_0^N \left(p_{*,j}(h)\,\tau_{*,j}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{\rho-1}{\rho}}}\right)$$

Introducing the specification of $\varphi'^{-1}(x)$ in (17):

$$s_{i,j}(h) = \frac{1}{\eta}\left(\left(\frac{\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{\rho-1}{\rho}} p_{i,j}(h)\,\tau_{i,j}}{\left(\int\limits_0^N \left(p_{*,j}(h)\,\tau_{*,j}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{\rho-1}{\rho}}}\right)^{\frac{1}{\rho-1}} + (\eta - 1)\right)$$

$$s_{i,j}(h) = \frac{\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{1}{\rho}}}{\eta}\left(\frac{p_{i,j}(h)\,\tau_{i,j}}{\left(\int\limits_0^N \left(p_{*,j}(h)\,\tau_{*,j}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{\rho-1}{\rho}}}\right)^{\frac{1}{\rho-1}} + \frac{(\eta-1)}{\eta}$$

Finally, introducing the parameter bundles ω_1 and ω_2 to simplify the notation gives the result in the manuscript.

$$s_{i,j}(h) = \omega_1 \frac{\left(p_{i,j}(h)\,\tau_{i,j}\right)^{\frac{1}{\rho-1}}}{\left(\int\limits_0^N \left(p_{*,j}(h)\,\tau_{*,j}\right)^{\frac{\rho}{\rho-1}} dh\right)^{\frac{1}{\rho}}} + \omega_2 \tag{20}$$

with

$$\omega_1 = \frac{\left(\eta + N(1-\eta)^{\rho}\right)^{\frac{1}{\rho}}}{\eta}$$

and

$$\omega_2 = \frac{\eta - 1}{\eta}$$

Manufacturing Price index G_i

The next step is to use this result to work out the manufacturing price index G_i, by inserting this result into the the general specification given by equation (8), where the share variable replaces $\varphi'^{-1}\left(\frac{p_{*,i}(h)\tau_{*,i}}{\tilde{P}_i}\right)$.

$$G_i = \int_0^N p_{*,i}(h)\,\tau_{*,i} s_{*,i}(h)\,dh$$

$$G_i = \int_0^N \left(\omega_1 \frac{(p_{*,i}(h)\,\tau_{*,i})^{\frac{\rho}{\rho-1}}}{\left(\int\limits_0^N (p_{*,i}(h)\,\tau_{*,i})^{\frac{\rho}{\rho-1}}\,dh \right)^{\frac{1}{\rho}}} + \omega_2 p_{*,i}(h)\,\tau_{*,i} \right) dh$$

Separating two elements of the integral sum and factoring out the constant terms:

$$G_i = \omega_1 \frac{\int_0^N (p_{*,i}(h)\,\tau_{*,i})^{\frac{\rho}{\rho-1}}\,dh}{\left(\int\limits_0^N (p_{*,i}(h)\,\tau_{*,i})^{\frac{\rho}{\rho-1}}\,dh \right)^{\frac{1}{\rho}}} + \omega_2 \int_0^N p_{*,i}(h)\,\tau_{*,i} dh$$

Combining the two integral sums gives the manufacturing price index G_i:

$$G_i = \omega_1 \left(\int_0^N \left((p_{*,i}(h)\,\tau_{*,i})^{\frac{\rho}{\rho-1}} \right) dh \right)^{\frac{\rho-1}{\rho}} + \omega_2 \int_0^N p_{*,i}(h)\,\tau_{*,i} dh \qquad (21)$$

Elasticity ϵ_i and the markup equation

The last step involves using the specification of the sub-utility function to obtain the elasticity of demand in equation (13), and therefore the mark-up of prices over marginal costs. This elasticity is given by:

$$\frac{1}{\varepsilon_{i,j}(h)} = s_{i,j}(h) \frac{\varphi''(s_{i,j}(h))}{\varphi'(s_{i,j}(h))}$$

Given the functional forms of $\varphi'(x)$ and $\varphi''(x)$ given by (15) and (16), this gives the following specification:

$$\frac{1}{\varepsilon_{i,j}(h)} = s_{i,j}(h) \frac{\eta\rho(\rho-1)(\eta s_{i,j}(h) - (\eta-1))^{\rho-2}}{\rho(\eta s_{i,j}(h) - (\eta-1))^{\rho-1}}$$

Simplifying ρ and the terms in brackets and inverting gives:

$$\varepsilon_{i,j}(h) = \frac{s_{i,j}(h)\,\eta - (\eta-1)}{s_{i,j}(h)\,\eta(\rho-1)}$$

Replacing $\rho - 1 = -1/\sigma$ and dividing numerator and denominator by η:

$$\varepsilon_{i,j}(h) = -\sigma \frac{s_{i,j}(h) - \omega_2}{s_{i,j}(h)}$$

$$\varepsilon_{i,j}(h) = -\sigma \left(1 - \frac{\omega_2}{s_{i,j}(h)}\right) \tag{22}$$

This implies the following mark-up equation.

$$p_{i,j}(h) = \frac{s_{i,j}(h) - \omega_2}{s_{i,j}(h)\rho - \omega_2} \beta w_i \tag{23}$$

Taking the derivative confirms the existence of a pro-competitive effect:

$$\frac{\partial p_i(h)}{\partial s_{i,j}(h)} = \frac{s_{i,j}(h)\rho - \omega_2 - \rho\left(s_{i,j}(h) - \omega_2\right)}{\left(s_{i,j}(h)\eta\rho - (\eta - 1)\right)^2} \beta w_i$$

$$\frac{\partial p_i(h)}{\partial s_{i,j}(h)} = \frac{s_{i,j}(h)\rho - \omega_2 - \rho s_{i,j}(h) + \rho\omega_2}{\left(s_{i,j}(h)\eta\rho - (\eta - 1)\right)^2} \beta w_i$$

$$\frac{\partial p_i(h)}{\partial s_{i,j}(h)} = \frac{\omega_2(\rho - 1)}{\left(s_{i,j}(h)\eta\rho - (\eta - 1)\right)^2} \beta w_i \geq 0$$

2 A variable elasticity of substitution NEG model

2.1 Closing the model and describing equilibrium

The labour market clears in both regions, so that the the amount of labour available L_i equals the demand from both sectors.

$$L_i = \int_0^N \left(\alpha + \beta \sum_j m_{i,j}(h)\right) dh \tag{24}$$

Apart from labour market clearing, an equilibrium is characterised by zero profits, saturation of the consumer budget constraint, and the mark-up equation.

$$\pi_i(h) = \sum_j p_{i,j}(h)\, m_{i,j}(h) - w_i \left(\alpha + \beta \sum_j m_{i,j}(h)\right)$$

$$\int_0^N p_{*,i}(h)\,\tau_{*,i} m_{*,i}(h)\,dh = w_i L_i$$

$$p_{i,j}(h) = \frac{s_{i,j}(h) - \omega_2}{s_{i,j}(h)\,\rho - \omega_2}\beta w_i$$

Proposition 1 establishes that firms producing in the same region will charge the same prices, and produce the same amounts, regardless of the variety produced. An equilibrium is therefore a set of variables $\{p_{i,j}; m_{i,j}; w_i; n_i\}$ that satisfy the system above.

2.2 Solving for symmetric equilibria

The two equilibria discussed in the paper, autarky and free trade, are symmetric and are therefore described by similar equations.

- In autarchy, firms only produce for their home region, and so the equilibrium is symmetric simply by virtue of Proposition 1. The only relevant transport cost is the home transport cost $\tau_{i,i} = 1$. The following equations characterise the equilibrium set of variables $\{p_{i,i}; m_{i,i}; w_i; n_i\}$
- If free trade occurs, i.e. $\forall i, j;\ \tau_{i,j} = 1$, Proposition 4 establishes, under the assumption of mill pricing, that there is a unique price for all flows, as well as a unique wage. The equations below then define the set of a aggregate variables $\{p; m; w; N\}$, where the number of firms in a given region and the output flows produced can be obtained by multiplying N or m by the share of overall labour that regions possesses.

For ease of notation, the variables in the following equations do not have subscripts, and can therefore be considered as the aggregate variable set $\{p; m; w; N\}$ of the free trade equilibrium. The solutions are nevertheless also applicable to each individual region i in the autarchic equilibrium.

Preliminary work: simplifying shares s

The share variable is given in (20) by:

$$s_{i,j}(h) = \omega_1 \frac{(p_{i,j}(h)\,\tau_{i,j})^{\frac{1}{\rho-1}}}{\left(\int\limits_0^N (p_{*,j}(h)\,\tau_{*,j})^{\frac{\rho}{\rho-1}}\, dh\right)^{\frac{1}{\rho}}} + \omega_2$$

In the symmetric equilibria considered, prices $p_{i,j}(h)$ are symmetric across varieties and target regions such that $\forall i, j, h;\ p_{i,j}(h) = p$, and furthermore transport costs are equal to one, so this simplifies down to:

$$s = \omega_1 \frac{p^{\frac{1}{\rho-1}}}{\left(Np^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} + \omega_2$$

This can be simplified further by eliminating prices:

$$s = \omega_1 \frac{1}{N^{\frac{1}{\rho}}} + \omega_2$$

At this point, one must remember that ω_1 depends on the number of varieties traded N.

$$s = \frac{(\eta + N(1-\eta)^\rho)^{\frac{1}{\rho}}}{\eta} \frac{1}{N^{\frac{1}{\rho}}} + \omega_2$$

$$s = \left(\frac{\eta^{1-\rho}}{N^{\frac{1}{\rho}}} + \frac{(1-\eta)^\rho}{\eta^\rho}\right)^{\frac{1}{\rho}} + \omega_2$$

$$s = \left(\frac{\eta^{1-\rho}}{N^{\frac{1}{\rho}}} + (-\omega_2)^\rho\right)^{\frac{1}{\rho}} + \omega_2 \tag{25}$$

Solving for wages w and quantities m

The zero-profit condition in a symmetric equilibrium is:

$$\pi = pm - w(\alpha + \beta m) = 0$$

Furthermore, the aggregate budget constraint of consumers is:

$$Npm = wL$$

Solving for wages:

$$w = \frac{Npm}{L} \tag{26}$$

Replacing the wages in the zero-profit condition gives:

$$pm - \frac{Npm}{L}(\alpha + \beta m) = 0$$

Solving the term in brackets for m:

$$m = \frac{1}{\beta}\left(\frac{L}{N} - \alpha\right) \tag{27}$$

Solving for prices p and firm mass n

Prices are given very simply the mark-up equation (23) and the share variable (25).

$$p = \frac{s - \omega_2}{s\rho - \omega_2}\beta w \tag{28}$$

The aggregate labour market clearing condition in a symmetric equilibrium is given by:

$$L = N(\alpha + \beta m)$$

Introducing (26) to substitute for m and rearranging gives:

$$N\left(\alpha + \beta\frac{wL}{Np}\right) = L$$

$$N\alpha + \beta\frac{wL}{p} = L$$

$$N = \frac{L}{\alpha}\left(1 - \beta\frac{w}{p}\right)$$

Using the price equation (28) to eliminate $\beta\frac{w}{p}$ and rearranging:

$$N = \frac{L}{\alpha}\left(1 - \frac{\rho s - \omega_2}{s - \omega_2}\right)$$

$$N = \frac{L}{\alpha}\left(\frac{(1-\rho)s}{s - \omega_2}\right)$$

The more intuitive notation used in the manuscript uses the elasticity of demand:

$$N = \frac{L}{-\alpha\varepsilon} \tag{29}$$

with

$$\varepsilon = -\sigma(1 - \frac{\omega_2}{s}) \tag{30}$$

Equations (29) (30) together with the share variable (25) form a system of equations that can be solved for s, ε and N. Proposition 3 shows that this equilibrium is unique.